1. (**20pts**) This question is about singular value decomposition.

(a) Consider

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

i. Compute $A^T A$. Find the eigenvalues of $A^T A$.

ii. Compute the singular value decomposition of A.

iii. Write A as a linear combination of eigen-images.

(b) Hence or otherwise, compute the singular value decomposition of

$$A' = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

and write A' as a linear combination of eigen-images.

(a) (i) $A^T A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\det(A^T A - \lambda I)$

$= (2-\lambda)^2(1-\lambda) - 4 + 4\lambda$

$= -1\lambda^3 + 5\lambda^2 - 8\lambda + 4 - 4 + 4\lambda$

$= -\lambda(\lambda-1)(\lambda-4)$

$\therefore \lambda_1 = 4, \quad \lambda_2 = 1, \quad \lambda_3 = 0$

$\sigma_1 = 2, \quad \sigma_2 = 1, \quad \sigma_3 = 0$

(ii). λ_1: $\begin{bmatrix} 2-4 & 2 & \\ 2 & 2-4 & \\ & & 1-4 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -1 & \\ & & 1 \\ & & \end{bmatrix}$

$\therefore v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

λ_2: $\begin{bmatrix} 2-1 & 2 & \\ 2 & 2-1 & \\ & & 1-1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_3 : \begin{bmatrix} 2-0 & 2 & \\ 2 & 2-0 & \\ & & 1-0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 & \\ & & 1 \\ & & \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$

So we get V.

Note $A^T = A \Rightarrow A = V \Sigma V^T$

$\therefore A = V \Sigma V^T$

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & & \\ & 1 & \\ & & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}^T$$

(iii)

$$= 2 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}$$

$$+ 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} + 0 \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}$$

$$+ 0 \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b).

$$\begin{bmatrix} 2 & 2 & \\ 2 & 2 & \\ & & 3 \end{bmatrix}$$

$$= 4\begin{bmatrix} 1/2 & 1/2 & \\ 1/2 & 1/2 & \\ & & 0 \end{bmatrix} + 3\begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}$$

$$+ 0\begin{bmatrix} 1/2 & -1/2 & \\ -1/2 & 1/2 & \\ & & 0 \end{bmatrix}$$

$$= V\begin{bmatrix} 4 & & \\ & 3 & \\ & & 0 \end{bmatrix} V^T$$

7. Consider a matrix $A \in M_{M\times N}(\mathbb{R})$, and let

$$A = U\Sigma V^T$$

be one of its singular value decompositions, such that $\sigma_{ii} \geq \sigma_{jj}$ whenever $i < j$.

(a) Show that the K-tuple $(\sigma_{11}, \sigma_{22}, \ldots, \sigma_{KK})$, where $K = \min\{M, N\}$, is uniquely determined.

(b) Show that if $\{\sigma_{ii} : i = 1, 2, \ldots, K\}$ are distinct *and nonzero*, then the *first* K columns of U and V are uniquely determined up to a change of sign. In other words, for each $i = 1, 2, \ldots, K$, there are exactly two choices of $(\vec{u}_i, \vec{v}_i)$; denoting one choice by $(\vec{u}, \vec{v})$, the other is given by $(-\vec{u}, -\vec{v})$.

(a)

$$\underset{\mathbb{R}^{M\times N}}{\Sigma} = \begin{cases} \begin{bmatrix} \sigma_1 & & & 0 & \cdots & 0 \\ & \sigma_2 & & \vdots & & \vdots \\ & & \ddots & & & \\ & & \sigma_K & 0 & \cdots & 0 \end{bmatrix}, & \text{if } M < N \\[2ex] \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_K \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, & \text{if } M > N \\[2ex] \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_K \end{bmatrix}, & \text{if } M = N \end{cases}$$

$$AA^T = U\Sigma\Sigma^T U^T$$

$$\underset{\mathbb{R}^{M\times M}}{\Sigma\Sigma^T} = \begin{cases} \begin{bmatrix} \sigma_1 & & & 0 & \cdots & 0 \\ & \sigma_2 & & \vdots & & \vdots \\ & & \ddots & & & \\ & & \sigma_K & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_K \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, & \text{if } M < N \\[2ex] \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_K \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} \sigma_1 & & & 0 & \cdots & 0 \\ & \sigma_2 & & \vdots & & \vdots \\ & & \ddots & & & \\ & & \sigma_K & 0 & \cdots & 0 \end{bmatrix}, & \text{if } M > N \\[2ex] \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_K \end{bmatrix}\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_K \end{bmatrix}, & \text{if } M = N \end{cases}$$

$$
= \begin{cases}
\begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_K^2 \end{bmatrix}, & \text{if } M < N \\[2ex]
\begin{bmatrix} \sigma_1^2 & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & \sigma_K^2 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}, & \text{if } M > N \\[2ex]
\begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_K^2 \end{bmatrix}, & \text{if } M = N
\end{cases}
$$

$$
\underbrace{\Sigma^T \Sigma}_{\in \mathbb{R}^{N \times N}} = \begin{cases}
\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_K \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & 0 & \cdots & 0 \\ & \sigma_2 & & & \vdots & & \vdots \\ & & \ddots & & & & \\ & & & \sigma_K & 0 & \cdots & 0 \end{bmatrix}, & \text{if } M < N \\[2ex]
\begin{bmatrix} \sigma_1 & & & & 0 & \cdots & 0 \\ & \sigma_2 & & & \vdots & & \vdots \\ & & \ddots & & & & \\ & & & \sigma_K & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_K \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, & \text{if } M > N \\[2ex]
\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_K \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_K \end{bmatrix}, & \text{if } M = N
\end{cases}
$$

$$
= \begin{cases}
\begin{bmatrix} \sigma_1^2 & & & 0 & \cdots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & \sigma_K^2 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}, & \text{if } M < N \\[2ex]
\begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_K^2 \end{bmatrix}, & \text{if } M > N \\[2ex]
\begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_K^2 \end{bmatrix}, & \text{if } M = N
\end{cases}
$$

$\therefore \{\sigma_i\}_{i=1}^{K}$ are square roots of the K larger eigenvalues of $A^T A$ and $A A^T$, in descending order,

$\therefore$ unique.

(6). If $\sigma_1 > \sigma_2 > \cdots > \sigma_K > 0$,

σ_i^2 is an eigenvalue of AA^T,
with dimension $= 1$

$\therefore$ Two unit eigenvectors, with a difference in sign. (u_i or $-u_i$)

σ_i^2 is an eigenvalue of A^TA,
with dimension $= 1$

$\therefore$ Two unit eigenvectors, with a difference in sign. (v_i or $-v_i$)

Choosing a valid SVD:

$$A = U \Sigma V^T$$

$$= \begin{cases} \begin{bmatrix} | & & | & & | \\ u_1 & \cdots & u_K & \cdots & u_M \\ | & & | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_K \\ & 0 & \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_K^T & - \end{bmatrix} & \text{if } M \geq N \\ \begin{bmatrix} | & & | \\ u_1 & \cdots & u_K \\ | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_K & \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_K^T & - \\ & \vdots & \\ - & v_N^T & - \end{bmatrix} & \text{if } M < N \end{cases}$$

$$= \begin{bmatrix} | & & | \\ u_1 & \cdots & u_K \\ | & & | \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_K \end{bmatrix} \begin{bmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_K^T & - \end{bmatrix}$$

$$= \sum_{i=1}^{K} \sigma_i \, \underline{\underline{u_i}} \, \underline{\underline{v_i}}^T = \sum_{i=1}^{K} \sigma_i \, (\underline{\underline{-u_i}}) (\underline{\underline{-v_i}})^T$$

>0, so ok

$$= \sum_{i=1}^{K} (-\sigma_i) (\underline{\underline{-u_i}}) (\underline{\underline{v_i}}^T) = \sum_{i=1}^{K} (-\sigma_i) (\underline{\underline{u_i}}) (\underline{\underline{-v_i}})^T$$

<0, not ok

Haar Transform

Since Images are in Matrix form,
finding good ways to represent images means
finding good ways to store the rows and columns.
i.e. change the basis.

Let $\vec{a} \in \mathbb{R}^n$, $\{\vec{b}_1, \cdots, \vec{b}_n\}$ orthonormal basis,
we know $\vec{a} = \beta_1 \vec{b}_1 + \cdots + \beta_n \vec{b}_n$
to find the coefficients,

$$\vec{b}_j \cdot \vec{a} = \beta_1 \vec{b}_j \cdot \vec{b}_1 + \cdots + \beta_j \vec{b}_j \cdot \vec{b}_j + \cdots + \beta_n \vec{b}_j \cdot \vec{b}_n$$
$$= \beta_j .$$

$$\therefore B^T A = \begin{bmatrix} - & \vec{b}_1 & - \\ & \vdots & \\ - & \vec{b}_n & - \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix}$$

is changing the basis of columns of A.

$$B^T A B = \left(\begin{bmatrix} - & \vec{b}_1 & - \\ & \vdots & \\ - & \vec{b}_n & - \end{bmatrix} \begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix} \right) \begin{bmatrix} | & & | \\ \vec{b}_1 & \cdots & \vec{b}_n \\ | & & | \end{bmatrix}$$

is changing the basis of rows of $B^T A$

Also, if both C, D has linearly independent columns,

$C^T A D =$

$$\begin{bmatrix} | & & | \\ \vec{c}_1 & \cdots & \vec{c}_n \\ | & & | \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} - & \vec{d}_1^T & - \\ & \vdots & \\ - & \vec{d}_n^T & - \end{bmatrix}$$

$$= \sum_i \sum_j a_{ij} \begin{bmatrix} | & & | \\ \vec{c}_1 & \cdots & \vec{c}_n \\ | & & | \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} - & \vec{d}_1^T & - \\ & \vdots & \\ - & \vec{d}_n^T & - \end{bmatrix}$$

$$= \sum_i \sum_j a_{ij} \underline{\underline{\vec{c}_i \vec{d}_j^T}}$$

a basis of Image Space
proof of LI in the last page.

We learnt SVD to do image compression, which can reduce $H \times W$ pixel values to $(H+W+1) \cdot K$, $K =$ no. of singular values.

But we may need more values to store an image in SVD form (for large K).
This is because we also need to store the col. of U and V.

So we may want to find some good basis to represent all images, e.g. Haar Transform.

2. **(20pts)** Recall that the 0-th Haar function is

$$H_0(t) = \begin{cases} 1 & \text{if } 0 \le t < 1 \\ 0 & \text{otherwise,} \end{cases}$$

and the other Haar functions are defined by

$$H_{2^p+n}(t) = \begin{cases} 2^{\frac{p}{2}} & \text{if } \frac{n}{2^p} \le t < \frac{n+0.5}{2^p} \\ -2^{\frac{p}{2}} & \text{if } \frac{n+0.5}{2^p} \le t < \frac{n+1}{2^p} \\ 0 & \text{otherwise} \end{cases}$$

for $p = 0, 1, 2, \cdots$ and $n = 0, 1, 2, \cdots, 2^p - 1$.

(a) Write down the Haar transform matrix for a 4×4 image, i.e. the matrix such that the Haar transform of f is HfH^T.

(b) Compute the Haar transform $\tilde{g}$ of the following 4×4 image

$$g = \begin{pmatrix} 5 & 3 & 0 & 0 \\ 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) $H = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

(b).

$$\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 3 & & \\ 9 & & & \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} 7 & 3/2 & & \\ 7 & 3/2 & & \\ -2\sqrt{2} & 3/\sqrt{2} & & \\ 0 & 0 & & \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} & 0 \\ 1/2 & 1/2 & -1/\sqrt{2} & 0 \\ 1/2 & -1/2 & 0 & 1/\sqrt{2} \\ 1/2 & -1/2 & 0 & -1/\sqrt{2} \end{bmatrix}^T$$

$$= \begin{bmatrix} 17/4 & 17/4 & 11/2\sqrt{2} & 0 \\ 17/4 & 17/4 & 11/2\sqrt{2} & 0 \\ -1/2\sqrt{2} & -11/2\sqrt{2} & -7/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C, D \subseteq \mathbb{R}^{n \times n}$

Suppose both $\{\vec{c}_i\}_{i=1}^n$, $\{\vec{d}_j\}_{j=1}^n$ are L.I.

(the columns of C and D)

Then $\sum_{i=1}^n \sum_{j=1}^n \alpha_{ij} \vec{c}_i \vec{d}_j^T = 0$ ← zero $n \times n$ matrix

$$\Rightarrow \sum_{i=1}^n \sum_{j=1}^n \alpha_{ij} \left[d_{1j} \vec{c}_i \,\middle|\, \cdots \,\middle|\, d_{nj} \vec{c}_i \right] = 0$$

$$\Rightarrow \left[\sum_{i=1}^n \left(\sum_{j=1}^n \alpha_{ij} d_{1j} \right) \vec{c}_i \,\middle|\, \cdots \,\middle|\, \sum_{i=1}^n \left(\sum_{j=1}^n \alpha_{ij} d_{nj} \right) \vec{c}_i \right] = 0$$

By L.I of $\{\vec{c}_i\}$,

$$\sum_{j=1}^n \alpha_{ij} d_{1j} = \sum_{j=1}^n \alpha_{ij} d_{2j} = \cdots = \sum_{j=1}^n \alpha_{ij} d_{nj} = 0 \quad \forall i$$

i.e. $$\begin{bmatrix} \sum_{j=1}^n \alpha_{ij} d_{1j} \\ \vdots \\ \sum_{j=1}^n \alpha_{ij} d_{nj} \end{bmatrix} = 0 \qquad \forall i$$

$$\Rightarrow \sum_{j=1}^n \alpha_{ij} \begin{bmatrix} d_{1j} \\ \vdots \\ d_{nj} \end{bmatrix} = 0 \qquad \forall i$$

$$\Rightarrow \sum_{j=1}^n \alpha_{ij} \vec{d}_j = 0 \quad \forall i$$

by L.I of $\{\vec{d}_j\}$, $\alpha_{ij} = 0 \quad \forall i, j$.

$\therefore \{\vec{c}_i \vec{d}_j^T\}_{i,j=1}^n$ are L.I.